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b-Functions of prehomogeneous vector spaces of classical, parabolic type

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Abstract

We consider prehomogeneous vector spaces of parabolic type for the classical complex Lie algebras, and compute the b-functions of several variables for their basic relative invariants. We also give a description of the b-functions in terms of lace diagrams.

1 Introduction

Let G be a complex reductive Lie group and $\mathfrak{g}$ its Lie algebra. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$, and take the Cartan subalgebra $\mathfrak{h}$, the root system Δ and the simple root system Φ such that $\mathfrak{h}$ and every simple root space are contained in $\mathfrak{p}$. Let Φ_I be the subset of Φ consisting of the simple roots $\alpha \in \Phi$ for which $\mathfrak{g}_{-\alpha} \subset \mathfrak{p}$, where $\mathfrak{g}_{-\alpha}$ denotes the root space of $-\alpha$. Define subspaces $\mathfrak{g}_j$ of $\mathfrak{g}$ for an integer j as the sum of the root spaces $\mathfrak{g}_\alpha$ ($\alpha \in \Delta \cup \{0\}$. we interpret the root space corresponding to zero as $\mathfrak{h}$), where $\alpha \in \Delta \cup \{0\}$ satisfies the following condition:

$$\sum_{\beta \in \Phi_I} c_\beta = j, \text{ where } \alpha = \sum_{\beta \in \Phi_I} c_\beta \beta + \sum_{\beta \in \Phi \setminus \Phi_I} c_\beta \beta.$$

Then we have

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j, \qquad \mathfrak{p} = \bigoplus_{j \geq 0} \mathfrak{g}_j.$$

Let G_0 be a subgroup of G corresponding to the Lie algebra $\mathfrak{g}_0$. Then it is known that $(G_0, \mathfrak{g}_1)$ is a prehomogeneous vector space, namely G_0 has an open orbit on $\mathfrak{g}_1$. This prehomogeneous vector space is said to be of parabolic type.

For a prehomogeneous vector space (G, V) a nonzero polynomial $f \in \mathbb{C}[V]$ is called a relative invariant if $f(gv) = \chi(g) f(v)$ for any $g \in G$ and $v \in V$, where χ is a one-dimensional representation (character) of G. A relative invariant is said to be basic if it is an irreducible polynomial, and it is known that every relative invariant is a product of powers of basic ones. In Section 2 we list all the prehomogeneous vector spaces of parabolic type for the classical Lie algebras, and determine the relative invariants.

When $f \in \mathbb{C}[V]$ is a relative invariant of a reductive prehomogeneous vector space (G, V), it is known that there exists a polynomial $b(s) \in \mathbb{Q}[s]$ such that

$$f^*(\partial).f^{s+1} = b(s) f^s,$$

where $f^*(\partial)$ is the constant coefficient differential operator obtained by substituting the partial differential operators to the variables in f (precisely this is correct only if the representation of G on V satisfies some condition. We omit the details in this note.), and a dot means the differentiation. The polynomial $b(s)$ is called the b-function of f. Furthermore suppose that $f_1, f_2, \ldots, f_p \in \mathbb{C}[V]$ are relative invariants of a reductive prehomogeneous vector space (G, V). It is known that there exists a polynomial $b_i(s_1, s_2, \ldots, s_p) \in \mathbb{Q}[s_1, s_2, \ldots, s_p]$ such that

$$f_i^*(\partial).f_1^{s_1} \cdots f_i^{s_i+1} \cdots f_p^{s_p} = b_i(s_1, \ldots, s_p) f_1^{s_1} \cdots f_i^{s_i} \cdots f_p^{s_p}.$$

The polynomial $b_i(s_1, \ldots, s_p)$ is called the b-function of several variables. Note that in papers [1, 2, 3] b-functions of several variables are defined in more general setup, and they can be recovered from our $b_i(s_1, \ldots, s_p)$. In Sections 3 and 4 we determine the b-function of several variables for the prehomogeneous vector spaces of classical, parabolic type in two cases. The remaining cases are explained in a forthcoming paper [6].

Among the results in this note Proposition 1 is already given by Sugiyama [3], and part of Theorem 3 is already given by Fumihiro Sato [1]. Remark that we have another proof for them using the Capelli identities of odd type [4, 5, 6]. In Sugiyama [3] b-functions of several variables are described in terms of lace diagrams. We give a similar description for other b-functions.

2 The classification of PVs of classical, parabolic type

In this section we determine the prehomogeneous vector spaces of parabolic type corresponding to classical complex Lie groups and their parabolic subgroups.

2.1 Type A

Define the Lie group G, its Lie algebra $\mathfrak{g}$, and its Cartan subalgebra $\mathfrak{h}$ as

$$G = GL_{n+1} = GL(n+1, \mathbb{C}), \qquad \mathfrak{g} = \mathfrak{gl}_{n+1}(\mathbb{C}), \qquad \mathfrak{h} = \sum_{i=1}^{n+1} \mathbb{C} E_{ii},$$

where E_{ii} is the matrix unit. Take the root system Δ, the simple root system Φ and its subset Φ_I as

$$\begin{aligned}
\Delta &= \{\epsilon_i - \epsilon_j \mid 1 \leq i, j \leq n+1,\ i \neq j\}, \\
\Phi &= \{\alpha_i := \epsilon_i - \epsilon_{i+1} \mid i = 1, 2, \ldots, n\}, \\
\Phi_I &= \{\alpha_{p_1}, \alpha_{p_2}, \ldots, \alpha_{p_k}\} \qquad (1 \leq p_1 < p_2 < \cdots < p_k \leq n).
\end{aligned}$$

Then the prehomogeneous vector space $(G_0, \mathfrak{g}_1)$ is given as

$$\begin{aligned}
G_0 &\cong GL_{m_0} \times GL_{m_1} \times \cdots \times GL_{m_k} \qquad (m_i = p_{i+1} - p_i,\ p_0 = 0,\ p_{k+1} = n+1) \\
\mathfrak{g}_1 &\cong \mathrm{Mat}_{m_0, m_1} \oplus \mathrm{Mat}_{m_1, m_2} \oplus \cdots \oplus \mathrm{Mat}_{m_{k-1}, m_k},
\end{aligned}$$

where $\mathrm{Mat}_{a,b}$ denotes the set of matrices of size $a \times b$, and the action is given as

$$(g_0, g_1, \ldots, g_k).(X_1, X_2, \ldots, X_k) = (g_0 X_1 g_1^{-1}, g_1 X_2 g_2^{-1}, \ldots, g_{k-1} X_k g_k^{-1}) \tag{1}$$

for $(g_0, g_1, \dots, g_k) \in G_0$ and $(X_1, X_2, \dots, X_k) \in \mathfrak{g}_1$. Note that $\mathfrak{g}_1$ is illustrated as

$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & X_1 & 0 & \cdots & 0 \\ \vdots & 0 & X_2 & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ \vdots & & & 0 & X_k \\ 0 & \cdots & & \cdots & 0 \end{pmatrix} \middle| X_i \in \mathrm{Mat}_{m_{i-1},m_i} \right\}.$$

Thus the basic relative invariants are

$$\det(X_c X_{c+1} \cdots X_d), \text{ where } 1 \le c \le d \le k,\ m_{c-1} = m_d,\ m_t > m_d\ (c \le t < d).$$

2.2 Type C

We use transposition with respect to the anti-diagonal so that we can place the positive root spaces above the diagonal. For $m \times n$ matrix X, define ${}^T X \in \mathrm{Mat}_{n,m}$ as

$$({}^T X)_{i,j} = X_{m+1-j,n+1-i}.$$

Define $J_n \in \mathrm{Mat}_{n,n}$ as

$$J_n = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & 1 & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Define the Lie group G, its Lie algebra $\mathfrak{g}$, and its Cartan subalgebra $\mathfrak{h}$ as

$$\begin{aligned} G = Sp_{2n}^T &= \left\{ g \in GL_{2n} \mid {}^t g \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} \right\} \\ &= \left\{ g \in GL_{2n} \mid {}^T g \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix} g = \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix} \right\}, \\ \mathfrak{g} &= \left\{ \begin{pmatrix} A & B \\ C & -{}^T A \end{pmatrix} \mid A, B, C \in \mathrm{Mat}_{n,n},\ {}^T B = B,\ {}^T C = C \right\}, \\ \mathfrak{h} &= \bigoplus_{i=1}^{n} \mathbb{C}(E_{i,i} - E_{2n-i+1,2n-i+1}). \end{aligned}$$

Note that the group Sp_{2n}^T is isomorphic to

$$Sp_{2n} := \left\{ g \in GL_{2n} \mid {}^t g \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \right\}.$$

Let ϵ_i $(1 \le i \le n)$ be the dual basis to $E_{i,i} - E_{2n-i+1,2n-i+1}$ $(1 \le i \le n)$. Then we can take the root system Δ and the simple root system Φ as

$$\begin{aligned} \Delta &= \{\pm(\epsilon_i - \epsilon_j) \mid 1 \le i < j \le n\} \cup \{\pm(\epsilon_i + \epsilon_j) \mid 1 \le i \le j \le n\}, \\ \Phi &= \{\alpha_i := \epsilon_i - \epsilon_{i+1} \mid i = 1, 2, \dots, n-1\} \cup \{\alpha_n := 2\epsilon_n\}. \end{aligned}$$

The positive root space $\mathfrak{g}_\alpha$ is as follows:

$$\mathfrak{g}_{\epsilon_i-\epsilon_j} = \mathbb{C}(E_{i,j} - E_{2n+1-j,2n+1-i}) \qquad (1 \le i < j \le n),$$
$$\mathfrak{g}_{\epsilon_i+\epsilon_j} = \mathbb{C}(E_{i,2n+1-j} + E_{j,2n+1-i}) \qquad (1 \le i \le j \le n).$$

The positive root spaces are on the upper diagonal part. For example $\mathfrak{g}_{2\epsilon_1} = \mathbb{C}E_{1,2n}$.

2.2.1 Type C (1)

Let Φ_I be a non-empty subset of the set of the simple roots Φ. We have to divide Type C into two cases according to whether $\alpha_n = 2\epsilon_n$ is contained in Φ_I or not to describe the prehomogeneous vector space and its basic relative invariants.

First we consider the case where $\alpha_n \notin \Phi_I$. Set

$$\Phi_I = \{\alpha_{p_1}, \alpha_{p_2}, \ldots, \alpha_{p_k}\} \qquad (1 \le p_1 < p_2 < \cdots < p_k < n).$$

Then the prehomogeneous vector space $(G_0, \mathfrak{g}_1)$ is given as

$$G_0 \cong GL_{q_1} \times GL_{q_2} \times \cdots \times GL_{q_k} \times Sp^T_{2n-2p_k} \quad (q_1 = p_1,\ q_2 = p_2 - p_1, \ldots, q_k = p_k - p_{k-1}),$$
$$\mathfrak{g}_1 \cong \mathrm{Mat}_{q_1,q_2} \oplus \mathrm{Mat}_{q_2,q_3} \oplus \cdots \oplus \mathrm{Mat}_{q_k,2n-2p_k},$$

and the action is given as

$$(g_1, g_2, \ldots, g_{k+1}).(X_1, X_2, \ldots, X_k) = (g_1 X_1 g_2^{-1}, g_2 X_2 g_3^{-1}, \ldots, g_k X_k g_{k+1}^{-1}) \tag{2}$$

for $(g_1, g_2, \ldots, g_{k+1}) \in G_0$ and $(X_1, X_2, \ldots, X_k) \in \mathfrak{g}_1$. Note that $\mathfrak{g}_1$ is illustrated as

$$\mathfrak{g}_1 = \left\{ \left(\begin{array}{cccc|ccccc} 0 & X_1 & & & & & & & \\ & \ddots & \ddots & & & & & & \\ & & 0 & X_k & X'_k & 0 & & & \\ & & & 0 & 0 & {}^T X'_k & & & \\ \hline & & & & 0 & -{}^T X_k & & & \\ & & & & & \ddots & \ddots & & \\ & & & & & & 0 & -{}^T X_1 & \\ & & & & & & & 0 & \end{array} \right) \;\middle|\; \begin{array}{c} X_i \in \mathrm{Mat}_{q_i,q_{i+1}} \\ (1 \le i \le k-1), \\ X_k, X'_k \in \mathrm{Mat}_{q_k,n-p_k} \end{array} \right\}. \tag{3}$$

The matrix X_k in (2) corresponds to the matrix $(X_k X'_k)$ in (3).

It is easy to show that this representation is equivalent to the following representation.

$$(G'_0, \mathfrak{g}'_1) = (GL_{q_1} \times \cdots \times GL_{q_k} \times Sp_{2n-2p_k}, \mathrm{Mat}_{q_1,q_2} \oplus \mathrm{Mat}_{q_2,q_3} \oplus \cdots \oplus \mathrm{Mat}_{q_k,2n-2p_k}),$$
$$(h_1, \ldots, h_{k+1}).(Y_1, \ldots, Y_k) = (h_1 Y_1 h_2^{-1}, h_2 Y_2 h_3^{-1}, \ldots, h_k Y_k h_{k+1}^{-1})$$

for $(h_1, h_2, \ldots, h_{k+1}) \in G'_0$ and $(Y_1, Y_2, \ldots, Y_k) \in \mathfrak{g}'_1$. We describe the basic relative invariants for this representation. Since the matrix $Y_c Y_{c+1} \cdots Y_k \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} {}^t(Y_c Y_{c+1} \cdots Y_k)$ is alternating, we have the following basic relative invariants:

$$\mathrm{pf}(Y_c \cdots Y_k \begin{pmatrix} 0 & 1_{n-p_k} \\ -1_{n-p_k} & 0 \end{pmatrix} {}^t(Y_c \cdots Y_k))$$
$$(1 \le c \le k,\ q_c\text{: even},\ q_t > q_c\ (c < t \le k+1)),$$
$$\det(Y_c Y_{c+1} \cdots Y_d) \qquad (1 \le c \le d \le k,\ q_c = q_{d+1},\ q_t > q_c\ (c < t \le d+1)),$$

where $q_{k+1} = 2n - 2p_k$. Note that the column size of Y_k is always even in this case.

2.2.2 Type C (2)

Next we consider the case where $\alpha_n \in \Phi_I$. Set

$$\Phi_I = \{\alpha_{p_1}, \alpha_{p_2}, \dots, \alpha_{p_k}, \alpha_{p_{k+1}}\} \qquad (1 \le p_1 < p_2 < \cdots < p_k < p_{k+1} = n).$$

Then the prehomogeneous vector space $(G_0, \mathfrak{g}_1)$ is given as

$$\begin{aligned} &G_0 \cong GL_{q_1} \times GL_{q_2} \times \cdots \times GL_{q_k} \times GL_{n-p_k} \quad (q_1 = p_1,\ q_2 = p_2 - p_1, \dots, q_k = p_k - p_{k-1}),\\ &\mathfrak{g}_1 \cong \mathrm{Mat}_{q_1,q_2} \oplus \mathrm{Mat}_{q_2,q_3} \oplus \cdots \oplus \mathrm{Mat}_{q_k,n-p_k} \oplus \mathrm{Sym}^T_{n-p_k},\end{aligned}$$

where $\mathrm{Sym}^T_{n-p_k}$ denotes the set of the symmetric matrices of size $n - p_k$ with respect to the anti-diagonal. The action is given as

$$(g_1, g_2, \dots, g_{k+1}).(X_1, X_2, \dots, X_k, S) = (g_1 X_1 g_2^{-1}, g_2 X_2 g_3^{-1}, \dots, g_k X_k g_{k+1}^{-1}, g_{k+1} S\, {}^T g_{k+1})$$

for $(g_1, g_2, \dots, g_{k+1}) \in G_0$ and $(X_1, X_2, \dots, X_k, S) \in \mathfrak{g}_1$. Note that $\mathfrak{g}_1$ is illustrated as

$$\mathfrak{g}_1 = \left\{ \left(\begin{array}{cccc|cccc} 0 & X_1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 0 & X_k & 0 & & & \\ & & & 0 & S & 0 & & \\ \hline & & & & 0 & -{}^T X_k & & \\ & & & & & \ddots & \ddots & \\ & & & & & & 0 & -{}^T X_1 \\ & & & & & & & 0 \end{array} \right) \;\middle|\; \begin{array}{l} X_i \in \mathrm{Mat}_{q_i,q_{i-1}} \\ \quad (1 \le i \le k-1), \\ X_k \in \mathrm{Mat}_{q_k,n-p_k}, \\ S \in \mathrm{Sym}^T_{n-p_k} \end{array} \right\}. \qquad (4)$$

It is easy to show that this representation is equivalent to the following representation.

$$\begin{aligned} &(G_0', \mathfrak{g}_1') \\ &\quad = (GL_{q_1} \times \cdots \times GL_{q_k} \times GL_{n-p_k}, \mathrm{Mat}_{q_1,q_2} \oplus \mathrm{Mat}_{q_2,q_3} \oplus \cdots \oplus \mathrm{Mat}_{q_k,n-p_k} \oplus \mathrm{Sym}_{n-p_k}),\\ &(h_1, \dots, h_{k+1}).(Y_1, \dots, Y_k, S) = (h_1 Y_1 h_2^{-1}, h_2 Y_2 h_3^{-1}, \dots, h_k Y_k h_{k+1}^{-1}, h_{k+1} S\, {}^t h_{k+1}) \end{aligned}$$

for $(h_1, h_2, \dots, h_{k+1}) \in G_0'$ and $(Y_1, Y_2, \dots, Y_k, S) \in \mathfrak{g}_1'$. We describe the basic relative invariants for this representation. Since the matrix $Y_c Y_{c+1} \cdots Y_k S\, {}^t(Y_c Y_{c+1} \cdots Y_k)$ is symmetric, we have the following basic relative invariants:

$$\begin{aligned} &\det(S) \text{ and } \det(Y_c \cdots Y_k S\, {}^t(Y_c \cdots Y_k)) \qquad (1 \le c \le k,\ q_t > q_c\ (c < t \le k),\ n - p_k > q_c),\\ &\det(Y_c Y_{c+1} \cdots Y_d) \qquad (1 \le c \le d \le k,\ q_c = q_{d+1},\ q_t > q_c\ (c < t \le d)). \end{aligned}$$

2.3 Type D

Define the Lie group G, its Lie algebra $\mathfrak{g}$, and its Cartan subalgebra $\mathfrak{h}$ as

$$\begin{aligned} G = O_{2n}^T &= \left\{ g \in GL_{2n} \mid {}^t g \begin{pmatrix} 0 & J_n \\ J_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & J_n \\ J_n & 0 \end{pmatrix} \right\} \\ &= \left\{ g \in GL_{2n} \mid {}^t g J_{2n} g = J_{2n} \right\} \\ &= \left\{ g \in GL_{2n} \mid {}^T g g = 1_{2n} \right\}, \\ \mathfrak{g} &= \left\{ \begin{pmatrix} A & B \\ C & -{}^T A \end{pmatrix} \mid A, B, C \in \mathrm{Mat}_{n,n},\ {}^T B = -B,\ {}^T C = -C \right\} \\ &= \left\{ X \in \mathrm{Mat}_{2n} \mid {}^T X = -X \right\}, \\ \mathfrak{h} &= \bigoplus_{i=1}^{n} \mathbb{C}(E_{i,i} - E_{2n-i+1,2n-i+1}). \end{aligned}$$

Note that the group O_{2n}^T is isomorphic to

$$O_{2n} := \left\{ g \in GL_{2n} \mid {}^t g g = 1_{2n} \right\}.$$

Let ϵ_i $(1 \leq i \leq n)$ be the dual basis to $E_{i,i} - E_{2n-i+1,2n-i+1}$ $(1 \leq i \leq n)$. Then we can take the root system Δ and the simple root system Φ as

$$\begin{aligned} \Delta &= \{\pm(\epsilon_i - \epsilon_j) \mid 1 \leq i < j \leq n\} \cup \{\pm(\epsilon_i + \epsilon_j) \mid 1 \leq i < j \leq n\}, \\ \Phi &= \{\alpha_i := \epsilon_i - \epsilon_{i+1} \mid i = 1, 2, \ldots, n-1\} \cup \{\alpha_n := \epsilon_{n-1} + \epsilon_n\}. \end{aligned}$$

The positive root space $\mathfrak{g}_\alpha$ is as follows:

$$\begin{aligned} \mathfrak{g}_{\epsilon_i - \epsilon_j} &= \mathbb{C}(E_{i,j} - E_{2n+1-j,2n+1-i}) && (1 \leq i < j \leq n), \\ \mathfrak{g}_{\epsilon_i + \epsilon_j} &= \mathbb{C}(E_{i,2n+1-j} - E_{j,2n+1-i}) && (1 \leq i < j \leq n). \end{aligned}$$

The positive root spaces are on the upper diagonal part. The anti-diagonal entries of $\mathfrak{g}$ are zero.

2.3.1 Type D (1)

Let Φ_I be a non-empty subset of the set of the simple roots Φ. We have to divide Type D into two cases according to whether $\alpha_n = \epsilon_{n-1} + \epsilon_n$ is contained in Φ_I or not to describe the prehomogeneous vector space and its basic relative invariants.

First we consider the case where $\alpha_n \notin \Phi_I$. Set

$$\Phi_I = \{\alpha_{p_1}, \alpha_{p_2}, \ldots, \alpha_{p_k}\} \qquad (1 \leq p_1 < p_2 < \cdots < p_k < n).$$

Then the prehomogeneous vector space $(G_0, \mathfrak{g}_1)$ is given as

$$\begin{aligned} G_0 &\cong GL_{q_1} \times GL_{q_2} \times \cdots \times GL_{q_k} \times O_{2n-2p_k}^T \quad (q_1 = p_1,\ q_2 = p_2 - p_1, \ldots, q_k = p_k - p_{k-1}), \\ \mathfrak{g}_1 &\cong \mathrm{Mat}_{q_1,q_2} \oplus \mathrm{Mat}_{q_2,q_3} \oplus \cdots \oplus \mathrm{Mat}_{q_k,2n-2p_k}, \end{aligned}$$

and the action is given as

$$(g_1, g_2, \ldots, g_{k+1}).(X_1, X_2, \ldots, X_k) = (g_1 X_1 g_2^{-1}, g_2 X_2 g_3^{-1}, \ldots, g_k X_k g_{k+1}^{-1}) \tag{5}$$

for $(g_1, g_2, \ldots, g_{k+1}) \in G_0$ and $(X_1, X_2, \ldots, X_k) \in \mathfrak{g}_1$. Note that $\mathfrak{g}_1$ is illustrated as

$$\mathfrak{g}_1 = \left\{ \left(\begin{array}{cccc|cccc} 0 & X_1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 0 & X_k & X'_k & 0 & & \\ & & & 0 & 0 & -{}^TX'_k & & \\ \hline & & & & 0 & -{}^TX_k & & \\ & & & & & \ddots & \ddots & \\ & & & & & & 0 & -{}^TX_1 \\ & & & & & & & 0 \end{array} \right) \;\middle|\; \begin{array}{c} X_i \in \mathrm{Mat}_{q_i,q_{i+1}} \\ (1 \leq i \leq k-1), \\ X_k, X'_k \in \mathrm{Mat}_{q_k,n-p_k} \end{array} \right\}. \qquad (6)$$

The matrix X_k in (5) corresponds to the matrix $(X_k X'_k)$ in (6). Note that this $\mathfrak{g}_1$ differs from (3) only at the sign of $-{}^TX'_k$.

It is easy to show that this representation is equivalent to the following representation.

$$\begin{aligned} &(G'_0, \mathfrak{g}'_1) = (GL_{q_1} \times \cdots \times GL_{q_k} \times O_{2n-2p_k}, \mathrm{Mat}_{q_1,q_2} \oplus \mathrm{Mat}_{q_2,q_3} \oplus \cdots \oplus \mathrm{Mat}_{q_k,2n-2p_k}), \\ &(h_1, \ldots, h_{k+1}).(Y_1, \ldots, Y_k) = (h_1 Y_1 h_2^{-1}, h_2 Y_2 h_3^{-1}, \ldots, h_k Y_k \, {}^th_{k+1}) \end{aligned}$$

for $(h_1, h_2, \ldots, h_{k+1}) \in G'_0$ and $(Y_1, Y_2, \ldots, Y_k) \in \mathfrak{g}'_1$. We describe the basic relative invariants for this representation.

$$\begin{aligned} &\det(Y_c \cdots Y_k \, {}^t(Y_c \cdots Y_k)) \qquad (1 \leq c \leq k,\ q_t > q_c\ (c < t \leq k+1)), \\ &\det(Y_c Y_{c+1} \cdots Y_d) \qquad (1 \leq c \leq d \leq k,\ q_c = q_{d+1},\ q_t > q_c\ (c < t \leq d)), \end{aligned}$$

where $q_{k+1} = 2n - 2p_k$. Note that the matrix $Y_c Y_{c+1} \cdots Y_k \, {}^t(Y_c Y_{c+1} \cdots Y_k)$ is symmetric, and that the column size of Y_k is always even in this case.

2.3.2 Type D (2)

Next we consider the case where $\alpha_n \in \Phi_I$. Set

$$\Phi_I = \{\alpha_{p_1}, \alpha_{p_2}, \ldots, \alpha_{p_k}, \alpha_{p_{k+1}}\} \qquad (1 \leq p_1 < p_2 < \cdots < p_k < p_{k+1} = n).$$

Then the prehomogeneous vector space $(G_0, \mathfrak{g}_1)$ is given as

$$\begin{aligned} &G_0 \cong GL_{q_1} \times GL_{q_2} \times \cdots \times GL_{q_k} \times GL_{n-p_k} \quad (q_1 = p_1,\ q_2 = p_2 - p_1, \ldots, q_k = p_k - p_{k-1}), \\ &\mathfrak{g}_1 \cong \mathrm{Mat}_{q_1,q_2} \oplus \mathrm{Mat}_{q_2,q_3} \oplus \cdots \oplus \mathrm{Mat}_{q_k,n-p_k} \oplus \mathrm{Alt}^T_{n-p_k}, \end{aligned}$$

where $\mathrm{Alt}^T_{n-p_k}$ denotes the set of alternating matrices of size $n - p_k$ with respect to the anti-diagonal. The action is given as

$$(g_1, g_2, \ldots, g_{k+1}).(X_1, X_2, \ldots, X_k, A) = (g_1 X_1 g_2^{-1}, g_2 X_2 g_3^{-1}, \ldots, g_k X_k g_{k+1}^{-1}, g_{k+1} A \, {}^Tg_{k+1})$$

for $(g_1, g_2, \ldots, g_{k+1}) \in G_0$ and $(X_1, X_2, \ldots, X_k, A) \in \mathfrak{g}_1$. Note that $\mathfrak{g}_1$ is illustrated as

$$\mathfrak{g}_1 = \left\{ \left(\begin{array}{ccccc|cccc} 0 & X_1 & & & & & & & \\ & \ddots & \ddots & & & & & & \\ & & 0 & X_k & & 0 & & & \\ & & & 0 & & A & 0 & & \\ \hline & & & & & 0 & -{}^TX_k & & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & 0 & -{}^TX_1 \\ & & & & & & & & 0 \end{array} \right) \;\middle|\; \begin{array}{l} X_i \in \mathrm{Mat}_{q_i,q_{i+1}} \\ \quad (1 \le i \le k-1), \\ X_k \in \mathrm{Mat}_{q_k, n-p_k}, \\ A \in \mathrm{Alt}^T_{n-p_k} \end{array} \right\}.$$

Note that this $\mathfrak{g}_1$ differs from (4) only at $A \in \mathrm{Alt}^T_{n-p_k}$.

It is easy to show that this representation is equivalent to the following representation.

$$\begin{aligned} &(G'_0, \mathfrak{g}'_1) \\ &\quad = (GL_{q_1} \times \cdots \times GL_{q_k} \times GL_{n-p_k}, \mathrm{Mat}_{q_1,q_2} \oplus \mathrm{Mat}_{q_2,q_3} \oplus \cdots \oplus \mathrm{Mat}_{q_k,n-p_k} \oplus \mathrm{Alt}_{n-p_k}), \\ &(h_1, \ldots, h_{k+1}).(Y_1, \ldots, Y_k, A) = (h_1 Y_1 h_2^{-1}, h_2 Y_2 h_3^{-1}, \ldots, h_k Y_k h_{k+1}^{-1}, h_{k+1} A\, {}^t h_{k+1}) \end{aligned}$$

for $(h_1, h_2, \ldots, h_{k+1}) \in G'_0$ and $(Y_1, Y_2, \ldots, Y_k, A) \in \mathfrak{g}'_1$. We describe the basic relative invariants for this representation. Since the matrix $Y_c Y_{c+1} \cdots Y_k A\, {}^t(Y_c Y_{c+1} \cdots Y_k)$ is alternating, we have the following basic relative invariants:

$$\begin{aligned} &\mathrm{pf}(A) \qquad (n - p_k\text{: even}), \\ &\mathrm{pf}(Y_c \cdots Y_k A\, {}^t(Y_c \cdots Y_k)) \qquad (1 \le c \le k,\ q_c\text{: even},\ q_t > q_c\ (c < t \le k),\ n - p_k > q_c), \\ &\det(Y_c Y_{c+1} \cdots Y_d) \qquad (1 \le c \le d \le k,\ q_c = q_{d+1},\ q_t > q_c\ (c < t \le d)). \end{aligned}$$

2.4 Type B

Define the Lie group G, its Lie algebra $\mathfrak{g}$, and its Cartan subalgebra $\mathfrak{h}$ as

$$\begin{aligned} G = O^T_{2n+1} &= \left\{ g \in GL_{2n+1} \mid {}^t g J_{2n+1} g = J_{2n+1} \right\} \\ &= \left\{ g \in GL_{2n+1} \mid {}^T g g = 1_{2n+1} \right\}, \\ \mathfrak{g} &= \left\{ X \in \mathrm{Mat}_{2n+1} \mid {}^T X = -X \right\}, \\ \mathfrak{h} &= \bigoplus_{i=1}^{n} \mathbb{C}(E_{i,i} - E_{2n-i+2,2n-i+2}). \end{aligned}$$

Note that the group O^T_{2n+1} is isomorphic to

$$O_{2n+1} := \left\{ g \in GL_{2n+1} \mid {}^t g g = 1_{2n+1} \right\}.$$

Let ϵ_i $(1 \le i \le n)$ be the dual basis to $E_{i,i} - E_{2n-i+2,2n-i+2}$ $(1 \le i \le n)$. Then we can take the root system Δ and the simple root system Φ as

$$\begin{aligned} \Delta &= \{\pm(\epsilon_i - \epsilon_j) \mid 1 \le i < j \le n\} \cup \{\pm(\epsilon_i + \epsilon_j) \mid 1 \le i < j \le n\} \cup \{\pm\epsilon_i \mid 1 \le i \le n\}, \\ \Phi &= \{\alpha_i := \epsilon_i - \epsilon_{i+1} \mid i = 1, 2, \ldots, n-1\} \cup \{\alpha_n := \epsilon_n\}. \end{aligned}$$

The positive root space $\mathfrak{g}_\alpha$ is as follows:

$$\begin{aligned}
\mathfrak{g}_{\epsilon_i-\epsilon_j} &= \mathbb{C}(E_{i,j} - E_{2n+2-j,2n+2-i}) && (1 \leq i < j \leq n), \\
\mathfrak{g}_{\epsilon_i+\epsilon_j} &= \mathbb{C}(E_{i,2n+2-j} - E_{j,2n+2-i}) && (1 \leq i < j \leq n), \\
\mathfrak{g}_{\epsilon_i} &= \mathbb{C}(E_{i,n+1} - E_{n+1,2n+2-i}) && (1 \leq i \leq n).
\end{aligned}$$

The positive root spaces are on the upper diagonal part, and the anti-diagonal entries of $\mathfrak{g}$ are zero.

2.4.1 Type B (1)

Let Φ_I be a non-empty subset of the set of the simple roots Φ. We have to divide Type B into two cases according to whether $\alpha_n = \epsilon_n$ is contained in Φ_I or not to describe the prehomogeneous vector space and its basic relative invariants.

First we consider the case where $\alpha_n \notin \Phi_I$. Set

$$\Phi_I = \{\alpha_{p_1}, \alpha_{p_2}, \dots, \alpha_{p_k}\} \qquad (1 \leq p_1 < p_2 < \cdots < p_k < n).$$

Then the prehomogeneous vector space $(G_0, \mathfrak{g}_1)$ is given as

$$\begin{aligned}
G_0 &\cong GL_{q_1} \times GL_{q_2} \times \cdots \times GL_{q_k} \times O^T_{2n+1-2p_k} \\
&\qquad\qquad\qquad (q_1 = p_1,\ q_2 = p_2 - p_1, \dots, q_k = p_k - p_{k-1}), \\
\mathfrak{g}_1 &\cong \mathrm{Mat}_{q_1,q_2} \oplus \mathrm{Mat}_{q_2,q_3} \oplus \cdots \oplus \mathrm{Mat}_{q_k,2n+1-2p_k},
\end{aligned}$$

and the action is given as

$$(g_1, g_2, \dots, g_{k+1}).(X_1, X_2, \dots, X_k) = (g_1 X_1 g_2^{-1}, g_2 X_2 g_3^{-1}, \dots, g_k X_k g_{k+1}^{-1}) \tag{7}$$

for $(g_1, g_2, \dots, g_{k+1}) \in G_0$ and $(X_1, X_2, \dots, X_k) \in \mathfrak{g}_1$. Note that $\mathfrak{g}_1$ is illustrated as

$$\mathfrak{g}_1 = \left\{ \left(\begin{array}{cccc|c|cccc}
0 & X_1 & & & & & & & \\
 & \ddots & \ddots & & & & & & \\
 & & 0 & X_k & X'_k & X''_k & & & \\
 & & & 0 & 0 & 0 & -{}^T X''_k & & \\
\hline
 & & & & 0 & 0 & -{}^T X'_k & & \\
\hline
 & & & & & 0 & -{}^T X_k & & \\
 & & & & & & \ddots & \ddots & \\
 & & & & & & & 0 & -{}^T X_1 \\
 & & & & & & & & 0
\end{array} \right) \;\middle|\; \begin{array}{l} X_i \in \mathrm{Mat}_{q_i,q_{i+1}} \\ \quad (1 \leq i \leq k-1), \\ X_k, X''_k \in \mathrm{Mat}_{q_k,n-p_k}, \\ X'_k \in \mathrm{Mat}_{q_k,1} \end{array} \right\}. \tag{8}$$

The matrix X_k in (7) corresponds to the matrix $(X_k X'_k X''_k)$ in (8).

It is easy to show that this representation is equivalent to the following representation.

$$\begin{aligned}
&(G'_0, \mathfrak{g}'_1) = (GL_{q_1} \times \cdots \times GL_{q_k} \times O_{2n+1-2p_k}, \mathrm{Mat}_{q_1,q_2} \oplus \mathrm{Mat}_{q_2,q_3} \oplus \cdots \oplus \mathrm{Mat}_{q_k,2n+1-2p_k}), \\
&(h_1, \dots, h_{k+1}).(Y_1, \dots, Y_k) = (h_1 Y_1 h_2^{-1}, h_2 Y_2 h_3^{-1}, \dots, h_k Y_k\, {}^t h_{k+1})
\end{aligned}$$

for $(h_1, h_2, \ldots, h_{k+1}) \in G_0'$ and $(Y_1, Y_2, \ldots, Y_k) \in \mathfrak{g}_1'$. We describe the basic relative invariants for this representation.

$$\det(Y_c \cdots Y_k\, {}^t(Y_c \cdots Y_k)) \qquad (1 \leq c \leq k,\ q_t > q_c\ (c < t \leq k+1)),$$
$$\det(Y_c Y_{c+1} \cdots Y_d) \qquad (1 \leq c \leq d \leq k,\ q_c = q_{d+1},\ q_t > q_c\ (c < t \leq d+1)),$$

where $q_{k+1} = 2n+1-2p_k$. Note that the matrix $Y_c Y_{c+1} \cdots Y_k\, {}^t(Y_c Y_{c+1} \cdots Y_k)$ is symmetric, and that the column size of Y_k is always odd and greater than or equal to three in this case.

2.4.2 Type B (2)

Next we consider the case where $\alpha_n \in \Phi_I$. Set

$$\Phi_I = \{\alpha_{p_1}, \alpha_{p_2}, \ldots, \alpha_{p_k}, \alpha_{p_{k+1}}\} \qquad (1 \leq p_1 < p_2 < \cdots < p_k < p_{k+1} = n).$$

Then the prehomogeneous vector space $(G_0, \mathfrak{g}_1)$ is given as

$$G_0 \cong GL_{q_1} \times GL_{q_2} \times \cdots \times GL_{q_k} \times GL_{n-p_k} \quad (q_1 = p_1,\ q_2 = p_2 - p_1, \ldots, q_k = p_k - p_{k-1}),$$
$$\mathfrak{g}_1 \cong \mathrm{Mat}_{q_1,q_2} \oplus \mathrm{Mat}_{q_2,q_3} \oplus \cdots \oplus \mathrm{Mat}_{q_k,n-p_k} \oplus \mathbb{C}^{n-p_k}.$$

The action is given as

$$(g_1, g_2, \ldots, g_{k+1}).(X_1, X_2, \ldots, X_k, v) = (g_1 X_1 g_2^{-1}, g_2 X_2 g_3^{-1}, \ldots, g_k X_k g_{k+1}^{-1}, g_{k+1} v)$$

for $(g_1, g_2, \ldots, g_{k+1}) \in G_0$ and $(X_1, X_2, \ldots, X_k, v) \in \mathfrak{g}_1$. Note that $\mathfrak{g}_1$ is illustrated as

$$\mathfrak{g}_1 = \left\{ \left(\begin{array}{cccc|c|cccc} 0 & X_1 & & & & & & & \\ & \ddots & \ddots & & & & & & \\ & & 0 & X_k & & & & & \\ & & & 0 & v & & & & \\ \hline & & & & 0 & -{}^T v & & & \\ \hline & & & & & 0 & -{}^T X_k & & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & 0 & -{}^T X_1 \\ & & & & & & & & 0 \end{array} \right) \middle| \begin{array}{l} X_i \in \mathrm{Mat}_{q_i,q_{i+1}} \\ \quad (1 \leq i \leq k-1), \\ X_k \in \mathrm{Mat}_{q_k,n-p_k}, \\ v \in \mathbb{C}^{n-p_k} \end{array} \right\}.$$

Since there is no transposition with respect to anti-diagonal in this representation, we describe the basic relative invariants for this representation. Similarly to the preceding cases the matrix $X_c X_{c+1} \cdots X_k v\, {}^t(X_c X_{c+1} \cdots X_k v)$ is symmetric, and it seems that the determinant of this symmetric matrix is a relative invariant. But the situation is slightly different. The size of the above symmetric matrix should be one if determinant is nonzero. Then the determinant factors into the square of $\det(X_c X_{c+1} \cdots X_k v)$. This means that the determinant of the above symmetric matrix is not basic. Actually this representation is just a special case of (1) in Type A. Thus we have the following basic relative invariants:

$$\det(X_c X_{c+1} \cdots X_d) \qquad (1 \leq c \leq d \leq k+1,\ q_c = q_{d+1},\ q_t > q_c\ (c < t \leq d)),$$

where $X_{k+1} = v$ and $q_{k+1} = n - p_k$.

Note that this case fills the remaining case of Type B (1), that is, in Type B (2) the column size of the last matrix of $\mathfrak{g}_1$ is one, which is odd and greater than or equal to three in Type B (1).

2.5 Summary

The prehomogeneous vector spaces of classical, parabolic type and their basic relative invariants are as follows. The b-functions of several variables for the first two cases are described in the following sections.

From Type A and Type B (2)

$$G_0 = GL_{m_0} \times GL_{m_1} \times \cdots \times GL_{m_k},$$
$$\mathfrak{g}_1 = \mathrm{Mat}_{m_0,m_1} \oplus \mathrm{Mat}_{m_1,m_2} \oplus \cdots \oplus \mathrm{Mat}_{m_{k-1},m_k},$$
$$(g_0, g_1, \cdots, g_k).(X_1, X_2, \cdots, X_k) = (g_0 X_1 g_1^{-1}, g_1 X_2 g_2^{-1}, \ldots, g_{k-1} X_k g_k^{-1}).$$

The basic relative invariants are

$$\det(X_c X_{c+1} \cdots X_d), \text{ where } 1 \le c \le d \le k,\ m_{c-1} = m_d,\ m_t > m_d\ (c \le t < d).$$

From Type C (1)

$$G_0 = GL_{q_1} \times GL_{q_2} \times \cdots \times GL_{q_k} \times Sp_{2q_{k+1}},$$
$$\mathfrak{g}_1 = \mathrm{Mat}_{q_1,q_2} \oplus \mathrm{Mat}_{q_2,q_3} \oplus \cdots \oplus \mathrm{Mat}_{q_k,2q_{k+1}},$$
$$(g_1, g_2, \ldots, g_{k+1}).(X_1, X_2, \ldots, X_k) = (g_1 X_1 g_2^{-1}, g_2 X_2 g_3^{-1}, \ldots, g_k X_k g_{k+1}^{-1}).$$

The basic relative invariants are

$$\mathrm{pf}(X_c \cdots X_k \begin{pmatrix} 0 & 1_{q_{k+1}} \\ -1_{q_{k+1}} & 0 \end{pmatrix} {}^t(X_c \cdots X_k))$$
$$(1 \le c \le k,\ q_c\text{: even},\ q_t > q_c\ (c < t \le k),\ 2q_{k+1} > q_c),$$
$$\det(X_c X_{c+1} \cdots X_d)$$
$$(1 \le c \le d \le k,\ q_c = q_{d+1}\ (\text{or } q_c = 2q_{k+1} \text{ if } d = k),\ q_t > q_c\ (c < t \le d)).$$

From Type C (2)

$$G_0 = GL_{q_1} \times GL_{q_2} \times \cdots \times GL_{q_{k+1}},$$
$$\mathfrak{g}_1 = \mathrm{Mat}_{q_1,q_2} \oplus \mathrm{Mat}_{q_2,q_3} \oplus \cdots \oplus \mathrm{Mat}_{q_k,q_{k+1}} \oplus \mathrm{Sym}_{q_{k+1}},$$
$$(g_1, g_2, \ldots, g_{k+1}).(X_1, X_2, \ldots, X_k, S) = (g_1 X_1 g_2^{-1}, g_2 X_2 g_3^{-1}, \ldots, g_k X_k g_{k+1}^{-1}, g_{k+1} S\, {}^t g_{k+1}).$$

The basic relative invariants are

$$\det(S) \text{ and } \det(X_c \cdots X_k S\, {}^t(X_c \cdots X_k)) \qquad (1 \le c \le k,\ q_t > q_c\ (c < t \le k+1)),$$
$$\det(X_c X_{c+1} \cdots X_d) \qquad (1 \le c \le d \le k,\ q_c = q_{d+1},\ q_t > q_c\ (c < t \le d)).$$

From Type D (1) and Type B (1)

$$G_0 = GL_{q_1} \times GL_{q_2} \times \cdots \times GL_{q_k} \times O_{q_{k+1}},$$
$$\mathfrak{g}_1 = \mathrm{Mat}_{q_1,q_2} \oplus \mathrm{Mat}_{q_2,q_3} \oplus \cdots \oplus \mathrm{Mat}_{q_k,q_{k+1}},$$
$$(g_1, g_2, \ldots, g_{k+1}).(X_1, X_2, \ldots, X_k) = (g_1 X_1 g_2^{-1}, g_2 X_2 g_3^{-1}, \ldots, g_k X_k g_{k+1}^{-1}).$$

The basic relative invariants are

$$\det(X_c \cdots X_k\, {}^t(X_c \cdots X_k)) \qquad (1 \le c \le k,\ q_t > q_c\ (c < t \le k+1)\ ),$$
$$\det(X_c X_{c+1} \cdots X_d) \qquad (1 \le c \le d \le k,\ q_c = q_{d+1},\ q_t > q_c\ (c < t \le d)).$$

From Type D (2)

$G_0 = GL_{q_1} \times GL_{q_2} \times \cdots \times GL_{q_k} \times GL_{q_{k+1}}$,
$\mathfrak{g}_1 = \mathrm{Mat}_{q_1,q_2} \oplus \mathrm{Mat}_{q_2,q_3} \oplus \cdots \oplus \mathrm{Mat}_{q_k,q_{k+1}} \oplus \mathrm{Alt}_{q_{k+1}}$,
$(g_1, g_2, \ldots, g_{k+1}).(X_1, X_2, \ldots, X_k, A) = (g_1 X_1 g_2^{-1}, g_2 X_2 g_3^{-1}, \ldots, g_k X_k g_{k+1}^{-1}, g_{k+1} A\, {}^t g_{k+1})$.

The basic relative invariants are

$$\begin{aligned}
&\mathrm{pf}(A) \quad (q_{k+1}\text{: even}),\\
&\mathrm{pf}(X_c \cdots X_k A\, {}^t(X_c \cdots X_k)) \quad (1 \le c \le k,\ q_c\text{: even},\ q_t > q_c\ (c < t \le k+1)),\\
&\det(X_c X_{c+1} \cdots X_d) \quad (1 \le c \le d \le k,\ q_c = q_{d+1},\ q_t > q_c\ (c < t \le d)).
\end{aligned}$$

3 b-Functions for Type A

In Sugiyama [3] b-functions of several variables are computed for Type A, and they are also described in terms of lace diagrams. We recall his result in this section. Remark that Sugiyama [3] has obtained b-functions for more general cases.

For a relative invariant f of a reductive prehomogeneous vector space let $f^*(\partial)$ be the constant coefficient differential operator obtained by substituting the partial differential operators to the variables. The b-function $b(s)$ of f is defined by

$$f^*(\partial).f^{s+1} = b(s) f^s$$

as mentioned in Introduction. Furthermore suppose that $f_1, f_2, \ldots, f_p \in \mathbb{C}[V]$ are relative invariants of a reductive prehomogeneous vector space (G, V). The b-function $b_i(s_1, s_2, \ldots, s_p)$ in $\mathbb{Q}[s_1, s_2, \ldots, s_p]$ of several variables is defined as

$$f_i^*(\partial).f_1^{s_1} \cdots f_i^{s_i+1} \cdots f_p^{s_p} = b_i(s_1, \ldots, s_p) f_1^{s_1} \cdots f_i^{s_i} \cdots f_p^{s_p}.$$

Set

$$\begin{aligned}
&G = GL_{m_0} \times GL_{m_1} \times \cdots \times GL_{m_k},\\
&V = \mathrm{Mat}_{m_0,m_1} \oplus \mathrm{Mat}_{m_1,m_2} \oplus \cdots \oplus \mathrm{Mat}_{m_{k-1},m_k},\\
&\quad (g_0, g_1, \cdots, g_k).(X_1, X_2, \cdots, X_k) = (g_0 X_1 g_1^{-1}, g_1 X_2 g_2^{-1}, \ldots, g_{k-1} X_k g_k^{-1}).
\end{aligned}$$

Then the basic relative invariants are

$$\det(X_c X_{c+1} \cdots X_d),\ \text{where } 1 \le c \le d \le k,\ m_{c-1} = m_d,\ m_t > m_d\ (c \le t < d).$$

Denote the basic relative invariants by $f_1, f_2, \ldots, f_p$ (in any order).

Proposition 1 (Sugiyama [3]). The b-function of several variables for (G, V) is

$$b_i(s_1, \ldots, s_p) = \prod_{c=c_i}^{d_i} \prod_{j=1}^{m_{d_i}} \left(\left(\sum_{\substack{1 \le l \le p,\\ f_l \supset X_c,\ m_{d_l} \ge j}} s_l \right) + m_c + 1 - j \right),$$

where

$$f_i = \det(X_{c_i} X_{c_i+1} \cdots X_{d_i}),$$

and $f_l \supset X_c$ means that X_c appears in the definition of f_l, that is, $c_l \le c \le d_l$. □

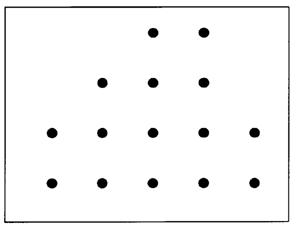
Remark that this proposition can be obtained also by the Capelli identity of odd type. This b-function can be described by lace diagrams. Take

$$G = GL_2 \times GL_3 \times GL_4 \times GL_4 \times GL_2,$$
$$V = \mathrm{Mat}_{2,3} \oplus \mathrm{Mat}_{3,4} \oplus \mathrm{Mat}_{4,4} \oplus \mathrm{Mat}_{4,2}$$

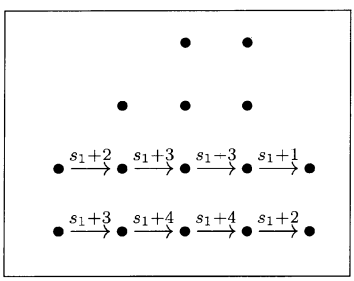
for example to explain the lace diagram. We have the basic relative invariants

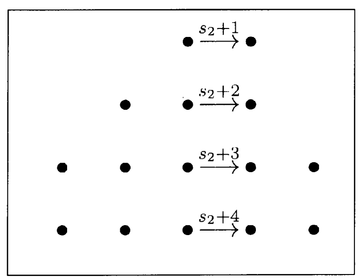
$$f_1 = \det(X_1X_2X_3X_4),$$
$$f_2 = \det(X_3),$$

where $(X_1, X_2, X_3, X_4) \in V$. First we draw dots according to the sizes of the groups (2, 3, 4, 4, 2).

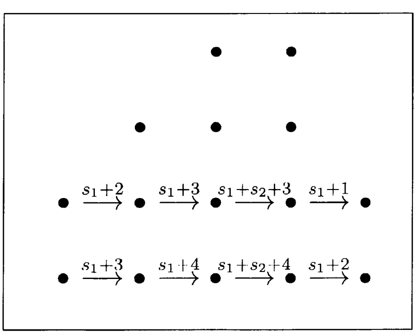


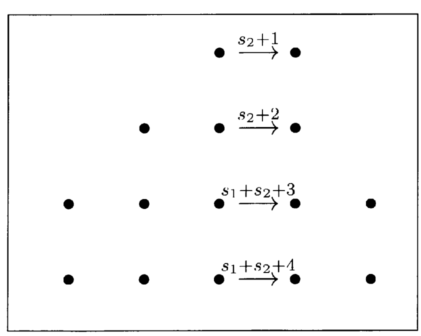
Next we draw horizontal arrows with linear factor. Arrows run from cth column to $(d+1)$st column if $f_i = \det(X_c \cdots X_d)$. The following are diagrams for f_1 and f_2.





We put linear factors $s_i + t$ to the arrows in the diagram of f_i. As to the constant term, t is determined so that the head of arrow is on tth dot from the top in its column. Finally in the diagram of f_i add s_j to the factor if f_i and f_j has arrows at the same position. Then we obtain the following diagrams for f_1 and f_2.





Proposition 2 (Sugiyama [3]). We have

$$b_i(s_1,\dots,s_p) = (\text{the product of factors in the lace diagram of } f_i). \quad \square$$

For the above example we have

$$\begin{aligned} b_1(s_1,s_2) &= (s_1+1)(s_1+2)^2(s_1+3)^2(s_1+4)(s_1+s_2+3)(s_1+s_2+4),\\ b_2(s_1,s_2) &= (s_1+s_2+3)(s_1+s_2+4)(s_2+1)(s_2+2). \end{aligned}$$

4 b-Functions for Type C (2)

Set

$$\begin{aligned} G &= GL_{m_0}\times GL_{m_1}\times\cdots\times GL_{m_k},\\ V &= \mathrm{Mat}_{m_0,m_1}\oplus\mathrm{Mat}_{m_1,m_2}\oplus\cdots\oplus\mathrm{Mat}_{m_{k-1},m_k}\oplus\mathrm{Sym}_{m_k};\\ &(g_0,g_1,\dots,g_k).(X_1,X_2,\dots,X_k,S) = (g_0X_1g_1^{-1}, g_1X_2g_2^{-1},\dots,g_{k-1}X_kg_k^{-1}, g_kS\,{}^tg_k). \end{aligned}$$

Then the basic relative invariants are

(i) $\det(S)$ and $\det(X_c\cdots X_kS\,{}^t(X_c\cdots X_k))$ $\quad (1\le c\le k;\ m_t > m_{c-1}\ (c\le t\le k))$,

(ii) $\det(X_cX_{c+1}\cdots X_d)$ $\quad (1\le c\le d\le k,\ m_{c-1}=m_d,\ m_t>m_{c-1}\ (c\le t<d))$.

Denote by $f_1, f_2,\dots,f_p$ the basic relative invariants of (G,V).

Theorem 3. Define λ_{ijl} as

$$\lambda_{ijl} = \begin{cases} 1 & (\ f_j\text{: type (i)},\ f_i\cap f_j\neq\emptyset,\ m_{c_j-1}\ge l\ )\\ \frac{1}{2} & (\ f_j\text{: type (ii)},\ f_i\cap f_j\neq\emptyset,\ m_{c_j-1}\ge l\ )\\ 0 & (\text{otherwise}) \end{cases}$$

where $f_i\cap f_j\neq\emptyset$ means that they have an arrow at the same position. Then we have the following.

(1) If f_i is of type (i), then the b-function of several variables is

$$b_i(s_1,\dots,s_p) = \prod_{c=c_i}^{k}\prod_{l=1}^{m_{c_i-1}}\left(\left(\sum_{j=1}^{p}\lambda_{ijl}s_j\right)+\frac{m_c+1-l}{2}\right)\times\prod_{c=c_i-1}^{k}\prod_{l=1}^{m_{c_i-1}}\left(\left(\sum_{j=1}^{p}\lambda_{ijl}s_j\right)+\frac{m_c+2-l}{2}\right)$$

up to scaling.

(2) If f_i is of type (ii), then the b-function of several variables is

$$b_i(s_1,\dots,s_p) = \prod_{c=c_i}^{d_i}\prod_{l=1}^{m_{c_i-1}}\left(\left(\sum_{j=1}^{p}\lambda_{ijl}s_j\right)+\frac{m_c+1-l}{2}\right)$$

up to scaling. $\square$

This theorem is proved by using the Capelli identities of odd type [6]. If $m_0, m_1, \dots, m_k$ is strictly increasing, then Fumihiro Sato [1] has obtained the b-functions, where all the basic relative invariants are of type (i).

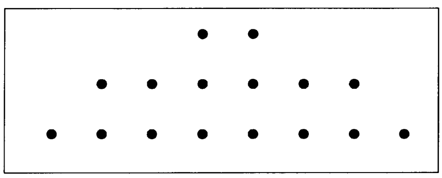
We can describe this b-function by the lace diagrams. Take

$$G = GL_1 \times GL_2 \times GL_2 \times GL_3,$$
$$V = \mathrm{Mat}_{1,2} \oplus \mathrm{Mat}_{2,2} \oplus \mathrm{Mat}_{2,3} \oplus \mathrm{Sym}_3$$

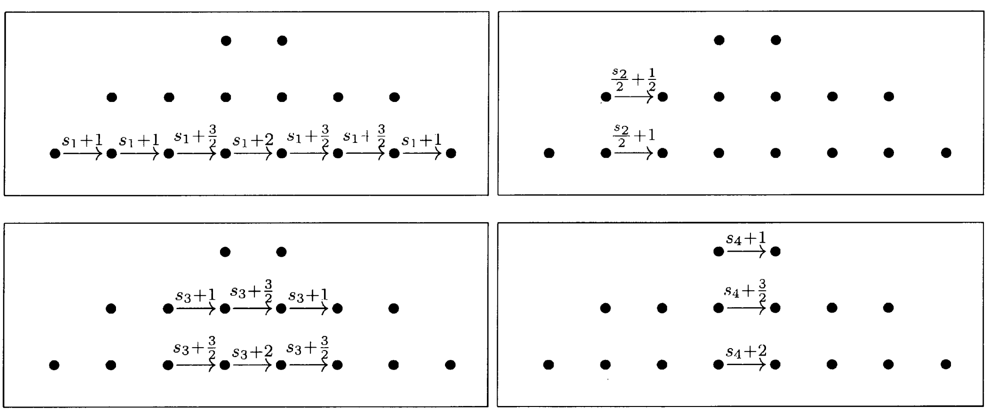
for example to explain the lace diagram. We have the basic relative invariants

$$f_1 = \det(X_1 X_2 X_3 S\, {}^tX_3\, {}^tX_2\, {}^tX_1),$$
$$f_2 = \det(X_2),$$
$$f_3 = \det(X_3 S\, {}^tX_3),$$
$$f_4 = \det(S),$$

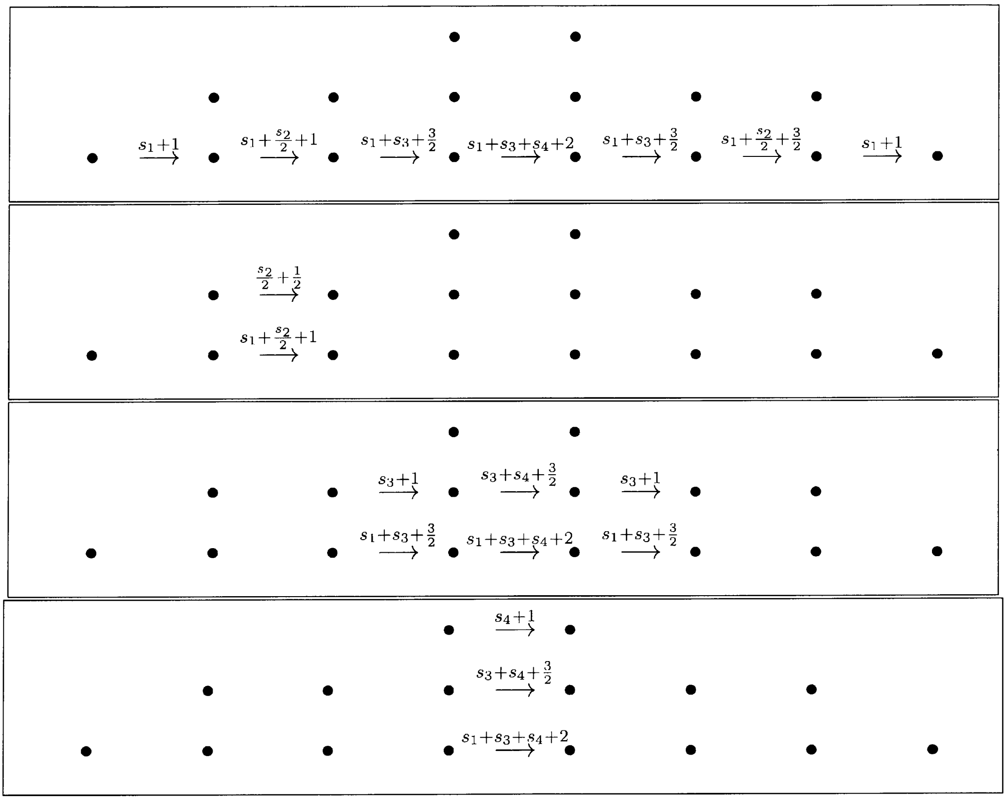
where $(X_1, X_2, X_3, S) \in V$. First we draw dots according to the sizes of the groups $(1, 2, 2, 3)$ followed by its reverse $(3, 2, 2, 1)$.



Next we draw arrows with linear factor. Arrows run as the matrices in the determinants represent linear maps. The following are diagrams for f_1, f_2, f_3, f_4.



We put linear factors $s_i + t$ to the arrows in the diagram of f_i of type (i) and put $s_i/2 + t$ for type (ii). As to the constant terms t of the factors $s_i + t$ or $s_i/2 + t$, if the head of the arrow is uth dot from the top, then $t = 1/2 + (u-1)/2$ on the left half of the diagram, and $t = 1 + (u-1)/2$ on the exact center and on the right half of the diagram. Finally in the diagram of f_i add s_j to the factor if f_i and f_j has arrows at the same position, and f_j is of type (i). Add $s_j/2$ to the factor if f_i and f_j has arrows at the same position, and f_j is of type (ii). In the second case add $s_j/2$ to the symmetric position on the right half. Then we obtain the following lace diagrams.



Theorem 4. We have

$$b_i(s_1, \dots, s_p) = (\text{the product of factors in the lace diagram of } f_i). \quad \square$$

b-Functions for the remaining prehomogeneous vector spaces of classical, parabolic type can be computed by using the Capelli identities of odd type, and they can be described by the lace diagrams. The details are explained in a forthcoming paper [6].

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